QCD inspired relativistic bound state model and meson structures

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A QCD inspired relativistic effective Hamiltonian model for the bound states of mesons has been constructed, which integrates the advantages of several QCD effective Hamiltonian models. Based on light-front QCD effective Hamiltonian model, the squared invariant mass operator of meson is used as the effective Hamiltonian. The model has been improved significantly in four major aspects: i) it is proved that in center of mass frame and in internal coordinate Hilbert subspace, the total angular momentum J of meson is conserved and the mass eigen equation can be expressed in total angular momentum representation and in terms of a set of coupled radial eigen equations for each J; ii) Based on lattice QCD results, a relativistic confining potential is introduced into the effective interaction and the excited states of mesons can be well described; iii) an SU(3) flavor mixing interaction is introduced phenomenologically to describe the flavor mixing mesons and the mass eigen equations contain the coupling among different flavor components; iv) the mass eigen equations are of relativistic covariance and the coupled radial mass eigen equations take full account of $L-S$ coupling and tensor interactions. The model has been applied to describe the whole meson spectra of about 265 mesons with available data, and the mass eigen equations have been solved nonperturbatively and numerically. The agreement of the calculated masses, squared radii, and decay constants with data is quite well. For the mesons whose mass data have large experimental uncertainty, the model produces certain mass values for test. For some mesons whose total angular momenta and parity are not assigned experimentally, the model gives a prediction of the spectroscopic configuration $^{2S+1}L_J$. The connection between our model and the recent low energy QCD issues-the infrared conformal scaling invariance and holographic QCD hadron models is discussed.

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I. INTRODUCTION

To study hadronic properties at low energy scales, nonperturbative effects must be taken into account[1]. To describe mesons and baryons, there are several main approaches: coupled Bethe-Salpeter(BS) and Dyson-Schwinger(DS) equation approach, relativistic constituent quark model based on Bethe-Salpeter equation(BSE), relativistic string Hamiltonian approach, and holographic light-front QCD approach. In the coupled Bethe-Salpeter and Dyson-Schwinger equation approach by P. Maris, P. Tandy, L. Kaptari et al.[2], the dressed quark propagators are assumed to have time like complex mass poles where the absence of real mass poles simulates quark confinement; the BS kernel is approximated by ladder rainbow truncation with two-parameter infrared structure. The approach contains four parameters in u-d-s quark sector and is consistent with quark and gluon confinement. Besides, it has the feature of preserving the relevant Ward identity and generating Dynamical chiral symmetry breaking. The vector mesons ρ, ϕ, and K^* are studied in detail, the calculated masses of ρ, ϕ,and K^* mesons and decay constants f_ρ, f_ϕ, and f_{K^*} are within 5% and 10% of the data respectively. Moreover, the ground-state spectra of light-quark mesons are also studied and a good description of flavor-octet pseudoscalar, vector, and axial-vector meson spectrum is obtained. The applicable domain of ladder truncation and the relative importance of various components of the two-body BS amplitude are also explored. However heavy quark mesons are not investigated and the number of mesons treated are not too many. R. Alkofer, P. Watson, and H. Weigel [3] follow the same approach, scalar and pseudoscalar, vector and axial vector mesons are studied. A similar approach is pursued by P. Jain and Munczek[4], about 50 mesons are investigated and the results are in good agreement with experiments. But heavy quarks are analyzed by non-relativistic dynamics. It should be noted that in contrary to Hamiltonian dynamics which works with wave functions that are not manifestly covariant quantities, the above BSE/DSE approaches emphasize the relativistic covariant aspect of the formalism and invariant quantities are studied.

The constituent quark model(CQM) works surprisingly well for most of the observed hadronic states [5, 6]. However, as a phenomenological theory, there are still some problems and puzzles that need to be clarified and understood[7]. One of the most important problems is relativistic effect. To solve the relativistic covariant problem of CQM, the relativistic constituent quark model based on Bethe-Salpeter equation was proposed by B.

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Metsch et al. in Bonn Group [8]. In this approach, the meson and baryon Hamiltonians are extracted from Bethe-Salpeter equation and the relativistic covariant constituent quark models for mesons and baryons are constructed. Based on Dirac structure of the two-body effective interactions, two types of models(A and B) are constructed. This approach addresses hadron mass spectra from ground state to 3GeV, light-flavor mesons, scalar excitations, linear Regge trajectory, pseudoscalar mixing, and parity doublet(for baryons). In this approach, the Dyson-Schwinger equation(DSE) is approximated by parametrization of infrared effective gluon propagator, the interaction kernel of BSE is given by single gluon exchange(OGE) and the confinement is parameterized by a string-like potential (having two versions defined by Dirac structures A and B). The instanton-induced spin-flavor dependent interaction is also included in the BSE kernel. The mass spectra up to 3 GeV, electroweak and strong-decay properties are calculated with 7 to 9 parameters. About 60 scalar and pseudo-scalar, vector and axial vector, and some tensor mesons with J=0,1,2 are calculated by models A and B, and compared to Godfrey-Isgur's calculation and experimental data (the deviation seems large but the errors are not indicated). Due to the Dirac structure of the effective interactions, spin-spin and spin-orbital interactions are included. Besides, heavy mesons are not treated.

The relativistic string Hamiltonian approach was proposed by A.M. Badalian et al.[9]. The merit of this approach is that the quark-anti-quark interaction and the confinement are generated by the relativistic string (through Nambu-Goto action for QCD vacuum fluctuation) which leads to a large reduction of the number of model parameters. After quantizing the action by path integral, they construct a Hamiltonian with a linear confining potential and hyperfine quark-anti-quark interactions. Using only one parameter of string tension, they study the systematic property of orbital excitations and rotation of mesons. The linear Regge trajectory relation between squared mass and orbital angular momentum is produced nicely and in agreement with the data for about 40 mesons. The relativistic string Hamiltonian approach is spin-independent. In the lowest order, this approach doesn't contain spin-spin, spin-orbital, and tensor interactions, thus it can produce the spin averaged mass spectra for mesons. However, to include the higher order effects by perturbation method, the hyperfine spin-dependent interactions could be obtained.

The holographic light-front QCD approach by S. J. Brodsky and G. F. de Teramond et al. [10] is based on light-front QCD and AdS/CFT correspondence. The AdS/CFT correspondence between string theory in AdS space and conformal field theories in physical space-time leads to an analytic, semi-classical model for strongly-coupled QCD, which has scale invariance and dimensional counting at short distances and color confinement at large distances. This correspondence also provides AdS/CFT or holographic QCD predictions for the analytic form of the frame-independent light-front wave functions (LFWFs) and masses of mesons and baryons. Recently, Brodsky *et al.*[10] have found that the transverse separation of quarks within hadron is related to holographic coordinate (the fifth dimensional z-coordinate) in AdS/CFT correspondence, the mass eigen equation of meson in light-front effective Hamiltonian approach corresponds to the equation of motion for the holographic field of effective gravity field of super string in AdS space at low energy limit. Recently, they have modified the gravitation background by using a positive-sign dilaton metric to generate confinement and break conformal symmetry. In the meanwhile, the chiral symmetry is broken and a mass scale is introduced to simulate the effect. Based on AdS/CFT correspondence, the holographic light-front QCD model yields a first order description of some hadronic spectra. This model is quite appealing and promising, since it has established a profound relationship between super string theory and QCD in low energy limit. In this model, very few parameters (cutoff parameter Λ_{QCD}) are used to obtain the spectra for both mesons and baryons, such as π, ρ, and Δ, etc., which fit the experimental data well[10]. However, for the large body of mesons, only few of them are described properly and a large part of mesons are still left over. Besides, in its preset form the full spin interactions are not treated properly although it has potential to describe spin splittings.

The light-front formalism[11] provides a convenient nonperturbative framework for the relativistic description of hadrons in terms of quark and gluon degrees of freedom[12]. Some fundamental nonperturbative light-front QCD approaches are available, such as light-front Bethe-Salpeter approach[13], holographic light-front QCD model[10], and light-front Hamiltonian method[14]. The light-front Bethe-Salpeter approach has been proposed by Kisslinger et al. to study pion form factor and the transition from non-perturbative to perturbative QCD calculation of pion form factor. Like the B-S approach of instant form, the equation of motion for light-front B-S wave function should be solved together with Schwinger-Dyson equation for dressed quark propagator, vertex, and self-energy, and the model parameters include confining potential strengths, and others for parametrization of the BS Kernel and the running quark masses. An interesting conclusion drawn from the study of this approach is that the perturbative QCD calculation works at the energy of 4-5 GeV, much lower than that explored previously by the instant form of QCD.

The effective light-front QCD Hamiltonian theory proposed by Brodsky and Pauli [14] is an attempt to describe the hadron structure as a bound constituent quark system in terms of Fock-space for the light-front wave-function. The effective Hamiltonian of the approach has been constructed recursively from the larger valence quark and anti-quark Fock sectors and reduced to the lowest valence quark-anti-quark sector[15]. Because of some unique features, particularly the apparent simplic-

ity of the light-front vacuum, this model is a promising approach to the bound-state problem of relativistic composite systems. Within the framework of the discretized light-front QCD, Pauli *et al.* have derived non-perturbatively an effective light-front Hamiltonian for mesons, which acts only on the $q\bar{q}$ sector[16, 17]. The mass eigen equations of mesons are formulated in momentum-helicity representation which hinders its solution in total angular momentum representation. Besides, in this effective Hamiltonian, confining potentials and flavor mixing interactions are lacking, so that the excited states of mesons and flavor diagonal light mesons can not be treated properly[18].

In order to apply the approach to describe mesons in whole $q\bar{q}$ sector, essential changes are needed. First we have proved that in center of mass frame (rest frame) and in internal coordinate Hilbert subspace, the total angular momentum of the meson system is conserved(see Appendix A and B). Then we are working in center of mass frame and in internal coordinate Hilbert subspace and make the following three significant improvements on the model: (1) transforming mass eigen equations from momentum-spin representation to total angular momentum representation and establishing a set of coupled radial mass eigen equations for each total angular momentum ; (2) introducing a relativistic confining potential into the effective meson interaction phenomenologically based on lattice QCD results ; (3) including an SU(3) flavor-mixing interaction in the model phenomenologically and obtaining a set of coupled radial eigen equations for different flavor components. In having done above, finally we have a complete QCD inspired relativistic bound state model for mesons on the whole $q\bar{q}$ sector. This model has been applied to about 265 mesons with available data and with total angular momentum from $J = 0$ to 6. The mass spectra, squared radii, and decay constants are calculated, and the calculated results are in good agreement with the data. While the most important physical results have been reported briefly in a short letter[19], the present article will provide detailed information and solid foundation of the model for completion

This paper is organized as follows. In Sec. II the QCD inspired relativistic bound state model for mesons is described and the relativistic mass eigen equations for bound states with any total angular momentum are derived. In Sec. III based on lattice QCD results, a relativistic confining potential in momentum space is introduced in the effective interaction of mesons. The effective interaction is extended to include an SU(3) flavor mixing interaction in Sec. IV. In Sec. V we present the numerical solutions for 265 mesons including both flavor-off and flavor diagonal mesons with $J = 0-6$. Sec.VI is an analysis of the results obtained. Finally, conclusion and discussion are given in Sec. VII. The four Appendices are for clarifying some important issues and for the derivation of key equations.

II. DESCRIPTION OF THE MODEL

For convenience, Brodsky and Pauli defined a light-front Lorentz invariant Hamiltonian [14]

$$H_{\mathrm{LC}} \equiv P^\mu P_\mu = P^- P^+ - \boldsymbol{P}_\perp^2 = \hat{M}_0^2, \tag{1}$$

The relativistic bound state problem in front form can be solved by solving the light-front mass eigen equation:

$$H_{\mathrm{LC}}|\Psi\rangle = M_0^2|\Psi\rangle \ . \tag{2}$$

If one disregards possible zero modes and works in the light-front gauge, this equation can be solved in terms of a complete set of Fock states $|\mu_n\rangle$:

$$\sum_{n'}\int d[\mu'_{n'}]\langle\mu_n|H_{\mathrm{LC}}|\mu'_{n'}\rangle\langle\mu'_{n'}|\Psi\rangle = M_0^2\langle\mu_n|\Psi\rangle \ . \tag{3}$$

For a meson, the ket $|\Psi\rangle$ holds:

$$\begin{aligned}|\Psi_{\mathrm{meson}}\rangle = &\sum_i \Psi_{q\bar{q}}(x_i,\vec{k}_{\perp i},\lambda_i)|q\bar{q}\rangle \\ &+\sum_i \Psi_{gg}(x_i,\vec{k}_{\perp i},\lambda_i)|gg\rangle \\ &+\sum_i \Psi_{q\bar{q}g}(x_i,\vec{k}_{\perp i},\lambda_i)|q\bar{q}g\rangle \\ &+\sum_i \Psi_{q\bar{q}q\bar{q}}(x_i,\vec{k}_{\perp i},\lambda_i)|q\bar{q}q\bar{q}\rangle \\ &+...\end{aligned} \tag{4}$$

Within the framework of discrete quantization of light-front QCD, infinite dimensional Fock space has been truncated at a proper cutoff energy and the energy truncation plays a role of renormalization in discrete light-front QCD. By Tamm-Dancoff projection method and resolvent technique, the equation of motion in a larger Fock space of multi-particles can be reduced to that in a smaller one with an effective interaction to account for the effect of the projected out part of the Fock space. The reduction and projection procedure can be carried out recursively, finally the effective Hamiltonian and its eigen equation on $q\bar{q}$ sector can be obtained. For flavor off-diagonal mesons, disregarding the zero modes and the two-gluon annihilation effect, Pauli et al. has obtained the effective mass eigen equation for mesons in light-front relative momentum coordinate space [16, 17]

$$\begin{aligned}&M_0^2\langle x,\vec{k}_\perp;\lambda_q,\lambda_{\bar{q}}|\psi\rangle = \\ &\quad\left[\frac{\overline{m}_q^2+\vec{k}_\perp^2}{x}+\frac{\overline{m}_{\bar{q}}^2+\vec{k}_\perp^2}{1-x}\right]\langle x,\vec{k}_\perp;\lambda_q,\lambda_{\bar{q}}|\psi\rangle \\ &\quad-\frac{4}{3}\frac{m_1m_2}{\pi^2}\sum_{\lambda'_q,\lambda'_{\bar{q}}}\int\frac{dx'd^2\vec{k}'_\perp\ R(x',k'_\perp)}{\sqrt{x(1-x)x'(1-x')}} \\ &\quad\frac{\overline{\alpha}(Q)}{Q^2}S_{\lambda_q\lambda_{\bar{q}};\lambda'_q\lambda'_{\bar{q}}}\langle x',\vec{k}'_\perp;\lambda'_q,\lambda'_{\bar{q}}|\psi\rangle \ ,\end{aligned} \tag{5}$$

This is a relativistic covariant mass eigen equation for mesons in center of mass fame and in internal Hilbert subspace. However the equation of motion is written in relative momentum and helicity representation, and the momentum-helicity plane wave function contains all possible components of partial waves of the spin spherical harmonic functions Φ_{JlsM}, the total angular momentum J and its z-component M are not conserved.

Despite this, Trittmann and Pauli[20] found an appropriate method which can calculate the eigenvalue spectrum separately for each $J_z = M$. To do so, they transformed the light-front coordinate x back to the coordinate k_3 by Terent'ev transformation[21], and used a unitary transformation to transform the Lepage-Brodsky spinors to the Bjorken-Drell spinors[22]. Then the mass eigen equation (5) becomes[23]:

$$\left[M_0^2 - (E_1(k) + E_2(k))^2\right] \varphi_{s_1 s_2}(\boldsymbol{k}) = \sum_{s_1' s_2'} \int d^3\boldsymbol{k'} U_{s_1 s_2; s_1' s_2'}(\boldsymbol{k}; \boldsymbol{k'}) \varphi_{s_1' s_2'}(\boldsymbol{k'}), \tag{6}$$

This integration equation is written in momentum-spin representation in terms of internal relative momenta of two quarks, spin singlet and triplet are mixed. For the same reason as discussed above, the momentum-spin plane wave does not conserve J and M. As noted in Ref[14], in general it is difficult to explicitly compute the total angular momentum of a bound state by using light-front quantization. However, as addressed in Introduction, in the center of mass frame and in internal Hilbert subspace, the total angular momentum is conserved. This makes it possible to solve the mass eigen equation in total angular momentum representation (see Appendices B,C,D).

Since in center of mass frame and in internal Hilbert subspace, the total angular momentum J^2 and J_z are conserved, we can transform the mass eigen equation (6) from momentum-spin representation to total angular momentum representation and establish the mass eigen equation for each J. Expanding the momentum-spin plane wave function in terms of the spin spherical harmonic functions $\Phi_{JslM}(\Omega_k, s_1, s_2)$ and projecting out the spin and angular part of the wave function in $|JslM\rangle$ subspace by the projecting operation,

$$\Big\langle \sum_{m\mu} \sum_{s_1 s_2} \langle lms\mu|JM\rangle \langle \frac{1}{2}s_1 \frac{1}{2}s_2|s\mu\rangle Y_{lm}(\Omega_k)\chi(s_1), \chi(s_2)\Big|, \tag{7}$$

we obtain the mass eigen equation for the radial wave function of $R_{Jsl}(k)$ (see Appendix C).

$$\left[M_0^2 - (E_1(k) + E_2(k))^2\right] R_{Jsl}(k) = \sum_{l'=|J-s'|}^{J+s'} \sum_{s'=0,1} \int k'^2 dk' U^J_{sl;s'l'}(k;k') R_{Js'l'}(k'). \tag{8}$$

This is a set of coupled equations for radial functions $R_{Jsl}(k)$ of different partial waves and of spin singlet and triplets, coupled by the tensor potential and by the relativistic spin-orbital potential. In this case, the eigen wave functions $R_{Jsl}(k)$ has the conventional definition and physical meaning. The bound states of mesons can be described concisely by the spectroscopic symbol of ${}^{2S+1}L_J$.

The kernel $U^J_{sl;s'l'}(k;k')$ can be written as (see Appendix D),

$$\begin{aligned} U^J_{sl;s'l'}(k;k') &= \sum_{mm'} \sum_{s_1 s_2} \sum_{s_1' s_2'} \int\int d\Omega_k d\Omega_{k'} \\ &\times \langle Y_{lm}(\Omega_k)|U_{s_1 s_2; s_1' s_2'}(\boldsymbol{k}, \boldsymbol{k'})|Y_{l'm'}(\Omega_{k'})\rangle \\ &\times \langle lms\mu|JM\rangle \langle \tfrac{1}{2}s_1 \tfrac{1}{2}s_2|s\mu\rangle \langle l'm's'\mu'|JM\rangle \langle \tfrac{1}{2}s_1' \tfrac{1}{2}s_2'|s'\mu'\rangle. \end{aligned} \tag{9}$$

The above kernel $U^J_{sl;s'l'}(k;k')$ contains different kinds of central potentials, relativistic spin-orbit coupling potentials, and tensor potentials changing l by $\Delta l = \pm 2$ and mixing spin singlet and triplets (see Appendix D).

III. INTRODUCING A CONFINING POTENTIAL

Quark confinement is one of the fundamental problems in QCD for hadronic physics. The confinement and the spontaneous breaking of chiral symmetry are key ingredients for solving the low-energy hadronic bound states from QCD, but none of them has been completely understood and solved. Numerical results show that the effective light-front Hamiltonian model proposed by Pauli et al. without confining potentials can well describe the ground states but can not apply to the radial excited states of mesons. To describe the excited states properly, the confining potential must be included in the model[18].

Fortunately, we can refer to the constituent quark model which is successful due to the inclusion of a phenomenological confining potential in some way[24]. The key idea of this model consists in the introduction of a linear confining potential in coordinate space based on the numerical calculations of lattice QCD, and this nonrelativistic confining potential can be generalized to relativistic form.

In nonrelativistic quark models the confining potential in configuration space is,

$$V_{\rm con}(r) = \lambda r + c \ , \tag{10}$$

where λ is the strength of the linear interaction, and c is a constant irrelevant in the present case and omitted hereafter. By Fourier transformation, the counterpart of the linear term λr in momentum space is obtained,

$$\begin{aligned} V_{\rm lin}(\boldsymbol{q}) &\sim -\frac{1}{|\boldsymbol{q}|^4} \ , \\ \boldsymbol{q} &= \boldsymbol{k} - \boldsymbol{k'} \ . \end{aligned} \tag{11}$$

At the point of $\boldsymbol{q} = 0$, the singularity indicates that the directly transformed result of linear potential could not

be described correctly in momentum space, which results in an ill-defined bound state equation[25]. However, some different methods were employed to solve this problem for the relativistic case. In the present paper, the correct form for $V_{\rm lin}(\boldsymbol{q})$ is constructed by introducing a small parameter η:

$$V(\boldsymbol{q}) \;=\; \lim_{\eta\to 0}\frac{\lambda}{2\pi^2}\frac{\partial^2}{\partial\eta^2}\left[\frac{1}{|\boldsymbol{q}|^2+\eta^2}\right] \tag{12}$$

The relativistic linear potential in momentum space $V_{\rm lin}(Q)$ is a direct generalization of the nonrelativistic one, just replacing the nonrelativistic $|\boldsymbol{q}|^2$ in (12) by the relativistic Q^2, which has the following specification in [6, 26],

$$Q^2 = (\mathbf{k}-\mathbf{k}')^2 + \varpi^2 \tag{13}$$

and

$$\varpi^2 = (E_1 - E_1')(E_2 - E_2') \tag{14}$$

Then the form of relativistic confining potential is,

$$V_{\rm con}(Q) \;=\; \lim_{\eta\to 0}\frac{\lambda}{2\pi^2}\frac{\partial^2}{\partial\eta^2}\left[\frac{1}{Q^2+\eta^2}\right] \tag{15}$$

Obviously, this confining potential is Lorentz covariant and can be used in either spin system or non-spin system.

Now as the relativistic confining potential $V_{\rm con}(Q)$ is included in the interaction, one has the new kernel $U_{s_1s_2;s_1's_2'}(\mathbf{k},\mathbf{k}')$,

$$U_{s_1s_2;s_1's_2'}(\mathbf{k},\mathbf{k}') = \tag{16}$$

$$\frac{4}{3}\frac{m_1m_2}{\pi^2}\sqrt{(\frac{1}{E_1}+\frac{1}{E_2})(\frac{1}{E_1'}+\frac{1}{E_2'})}\,\bar{u}(\mathbf{k},s_1)\bar{u}(-\mathbf{k},s_2)$$
$$\times[\gamma_\mu^{(1)}\cdot\gamma^{(2)\mu}V_V + \mathbb{I}^{(1)}\cdot\mathbb{I}^{(2)}V_S]u(\mathbf{k}',s_1')u(-\mathbf{k}',s_2') \tag{17}$$

The scalar and vector interaction potentials read

$$V_V \;=\; -\frac{\bar{\alpha}(Q)}{Q^2} - \frac{3}{4}\epsilon\, V_{\rm con}(Q)$$
$$V_S \;=\; -\frac{3}{4}(1-\epsilon)V_{\rm con}(Q) \tag{18}$$

where ϵ represents the scalar-vector mixing of the confining potential.

IV. INCLUDING A FLAVOR MIXING INTERACTION

It is extremely difficult to derive a simple form of flavor mixing interaction in the above effective Hamiltonian from light-front QCD at present. However, without flavor mixing potential, one can not deal with the flavor diagonal mesons such as π^0, ρ^0, and f_0, etc. In the fundamental hadronic theory, the quarks of u, d, and s have an approximate $SU(3)$ symmetry. Due to this symmetry, the quarks fields transform each other under the $SU(3)$ transformation [27],

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \longrightarrow exp\;[i\sum_a(\theta_a^V T_a + \theta_a^A T_a\gamma_5)]\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

where T_a are Gell-Mann Matrices. For convenience of numerical calculation, we introduce phenomenologically a simple flavor mixing interaction as follows,

$$\begin{aligned} V_f =&\gamma_0\left[T_{ud}^+(1)T_{ud}^+(2) + T_{ud}^-(1)T_{ud}^-(2)\right] \\ &+\delta_0\left[T_{us}^+(1)T_{us}^+(2) + T_{us}^-(1)T_{us}^-(2)\right. \\ &\left.+\,T_{ds}^+(1)T_{ds}^+(2) + T_{ds}^-(1)T_{ds}^-(2)\right] \end{aligned} \tag{19}$$

where γ_0 and δ_0 are the strengths of flavor-mixing interaction, the index 1 and 2 denote the quark and anti-quark in meson, respectively. The flavor SU(3) wave functions and generators are defined as,

$$|u\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |\bar{u}\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}; |d\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{20}$$

$$|\bar{d}\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |s\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}\; |\bar{s}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{21}$$

$$T_{ud}^+ = (T_{ud}^-)^\dagger = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \tag{22}$$

$$T_{us}^+ = (T_{us}^-)^\dagger = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} \tag{23}$$

$$T_{ds}^+ = (T_{ds}^-)^\dagger = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \tag{24}$$

The action of the flavor mixing interaction on flavor wave function is as follows,

$$V_f|u\bar{u}\rangle = \gamma_0|d\bar{d}\rangle + \delta_0|s\bar{s}\rangle \tag{25}$$
$$V_f|d\bar{d}\rangle = \gamma_0|u\bar{u}\rangle + \delta_0|s\bar{s}\rangle \tag{26}$$
$$V_f|s\bar{s}\rangle = \delta_0|d\bar{d}\rangle + \delta_0|u\bar{u}\rangle \tag{27}$$

Combining this interaction with the precious one in equations (18) , we have a set of flavor-coupled radial eigen equations for the flavor components of up, down, and strange quarks,

$$\left[M_0^2 - (E_1(k)+E_2(k))^2\right]R_{Jsl}^{p_1p_2}(k) \tag{28}$$
$$= \sum_{l'=|J-s'|}^{J+s'}\sum_{s'=0,1}\sum_{p_1'p_2'}\int k'^2dk'U_{Jsl;s'l'}^{p_1p_2,p_1'p_2'}(k;k')R_{Js'l'}^{p_1'p_2'}(k').$$

The interaction kernel including the flavor-mixing interaction is

$$U_{Jsl;s'l'}^{p_1p_2;p_1'p_2'}(k,k') = \sum_{mm'}\sum_{\mu\mu'}\sum_{s_1s_2}\sum_{s_1's_2'}\int\int d\Omega_k\Omega_{k'}$$

$$\times Y^*_{lm}(\Omega_k) W^{p_1p_2;p_1'p_2'}_{s_1s_2;s_1's_2'}(\mathbf{k},\mathbf{k}') Y_{l'm'}(\Omega_{k'}) \tag{29}$$
$$\langle lms\mu|JM\rangle\langle\frac{1}{2}s_1\frac{1}{2}s_2|s\mu\rangle\langle l'm's'\mu'|JM\rangle\langle\frac{1}{2}s_1'\frac{1}{2}s_2'|s'\mu'\rangle,$$

where $W^{p_1p_2;p_1'p_2'}_{s_1s_2;s_1's_2'}(\mathbf{k},\mathbf{k}')$ is defined as

$$W^{p_1p_2;p_1'p_2'}_{s_1s_2;s_1's_2'}(\mathbf{k},\mathbf{k}') = \frac{4}{3}\frac{m_{p_1}m_{p_2}}{\pi^2}\sqrt{\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\left(\frac{1}{E_1'}+\frac{1}{E_2'}\right)}\times$$
$$\bar{u}(p_1,\mathbf{k},s_1)\bar{u}(p_2,-\mathbf{k},s_2)\left[\gamma^\mu(p_1)\cdot\gamma_\mu(p_2)V_V + I^{(1)}\cdot I^{(2)}V_S\right]\times$$
$$\left[I_f + V_f\right]u'(p_1',\mathbf{k}',s_1')u'(p_2',-\mathbf{k}',s_2'). \tag{30}$$

where $p_1p_2, p_1'p_2' = \{u\bar{u}, d\bar{d}, s\bar{s}\}$, I_f is the identity operator in flavor space, V_V and V_S are the vector potential and scalar potential, respectively. For the flavor mixing mesons with total angular momentum J, the mass eigen equations are described explicitly by the following set of flavor-coupled equations

$$\left[M_0^2 - \left(E_u(k)+E_{\bar{u}}(k)\right)^2\right]R^{u\bar{u}}_{Jsl}(k) \tag{31a}$$
$$= \sum_{l'=|J-s'|}^{J+s'}\sum_{s'=0,1}\int k'^2dk'\left(U^{u\bar{u};u\bar{u}}_{Jsl;s'l'}(k,k')R^{u\bar{u}}_{Js'l'}(k,k') + \right.$$
$$\left.\gamma_0 U^{u\bar{u};d\bar{d}}_{Jsl;s'l'}(k,k')R^{d\bar{d}}_{Js'l'}(k,k') + \delta_0 U^{u\bar{u};s\bar{s}}_{Jsl;s'l'}(k,k')R^{s\bar{s}}_{Js'l'}(k,k')\right)$$
$$\left[M_0^2 - \left(E_d(k)+E_{\bar{d}}(k)\right)^2\right]R^{d\bar{d}}_{Jsl}(k) \tag{31b}$$
$$= \sum_{l'=|J-s'|}^{J+s'}\sum_{s'=0,1}\int k'^2dk'\left(\gamma_0 U^{d\bar{d};u\bar{u}}_{Jsl;s'l'}(k,k')R^{u\bar{u}}_{Js'l'}(k,k') + \right.$$
$$\left.U^{d\bar{d};d\bar{d}}_{Jsl;s'l'}(k,k')R^{d\bar{d}}_{Js'l'}(k,k') + \delta_0 U^{d\bar{d};s\bar{s}}_{Jsl;s'l'}(k,k')R^{s\bar{s}}_{Js'l'}(k,k')\right)$$
$$\left[M_0^2 - \left(E_s(k)+E_{\bar{s}}(k)\right)^2\right]R^{s\bar{s}}_{Jsl}(k) \tag{31c}$$
$$= \sum_{l'=|J-s'|}^{J+s'}\sum_{s'=0,1}\int k'^2dk'\left(\delta_0 U^{s\bar{s};u\bar{u}}_{Jsl;s'l'}(k,k')R^{u\bar{u}}_{Js'l'}(k,k') + \right.$$
$$\left.\delta_0 U^{s\bar{s};d\bar{d}}_{Jsl;s'l'}(k,k')R^{d\bar{d}}_{Js'l'}(k,k') + U^{s\bar{s};s\bar{s}}_{Jsl;s'l'}(k,k')R^{s\bar{s}}_{Js'l'}(k,k')\right)$$

V. NUMERICAL SOLUTIONS

In the above equations, J, s, l denote total angular momentum, total spin, and total orbital angular momentum, respectively. Different mesons can be classified by the spectroscopic symbol $^{2S+1}L_J$(or their combination), which is equivalent to the symbol J^{PC}. The space parity and charge conjugation parity are denoted as,

$$\begin{aligned} P &= (-1)^{L+1} \\ C &= (-1)^{L+S}. \end{aligned} \tag{32}$$

In the present model, the mass eigen value problem of mesons is described by a set of coupled integration equations, and the interaction includes a quark-anti-quark one gluon exchange potential V_{OGE} , a confining potential V_{con}, and a flavor mixing interaction V_f. If the confining potential has a pure iso-scalar structure($\epsilon = 0$) and the flavor mixing interaction is omitted, the interaction contains two parameters: the effective coupling constant $\bar{\alpha}$ and the confining potential strength λ. Besides, the flavor mixing interaction has two parameters, and the constituent quark masses are also indispensable parameters to describe spontaneous chiral symmetry breaking.

The numerical solution of the eigen equations can be obtained by discretization of integration equation (8) or $(31a-c)$, and the integration equations are transformed into matrix equations. The 4-fold integral of the kennel is completed by the integration technique of spherical harmonic functions and the angular momentum algebra, while the integration over k is performed by using Gauss-Legendre quadratures. The integration region $k \in [0,\infty)$ is projected onto the finite interval $x \in [-1,1]$ by $x = \frac{k-1}{k+1}$. The radial mass eigen equation (8) is discretized as follows:

$$\left[M_0^2 - \left(E_1(k_i)+E_2(k_i)\right)^2\right]R_{Jsl}(k_i) \tag{33}$$
$$= \sum_{l'=|J-s'|}^{J+s'}\sum_{s'=0,1}\sum_{j=1}^{N}U^J_{sl;s'l'}(k_i;k_j)R_{Js'l'}(k_j)k_j^2w_j \ ,$$

where w_j is the weight of integration. Before diagonalizing this matrix equations, special care should be taken for two kinds of singularities: the singularity at infinite k and the singularity as $k = k'$ inside the region of integration. The first one has been solved by the projection of the region $k \in [0,\infty)$ onto the finite interval $x \in [-1,1]$, and the second one is treated by infrared singularity treatment. The detailed procedures of calculation can be found in Ref.[18, 28].

The parameters of the model are determined from best fit to experimental data. In this paper, a purely scalar confining potential($\epsilon = 0$) is used. Reproducing the masses of π^0, $\pi^\pm$, and $\pi(1300)$, we can determine $\bar{\alpha}$, λ, and the masses of up and down quarks. Then by reproducing the masses of $K^\pm$, D^0, and $B^\pm$, the mass parameters of strange, charm, and bottom quarks are obtained. The parameters of flavor mixing interaction are determined by the best fit to the data of flavor diagonal mesons. From all the available data of mesons [29] with $J = 0-6$ (12 mesons are left for future study: including 6 exotic mesons and 6 mesons without any information about their J, s, L), we have obtained an appropriate set of 6 parameters for flavor off-diagonal mesons: $\bar{\alpha} = 0.2574$, $\lambda = 0.92\times10^4 MeV^2$, $m_{u/d} = 0.297GeV, m_s = 0.418GeV, m_c = 1.353GeV, m_b = 4.447GeV$; and for the flavor diagonal mesons: $\gamma_0 = 0.1$ and $\delta_0 = 0.1$. The number of the model parameters is minimum for this kind of semi-phenomenological models and comparable to BSE and CQM. The masses and wave functions of scalar and pseudoscalar, vector and axial-vector, tensor and pseudotensor mesons, and others with $J = 3-6$ have been calculated and compared with the experimen-

tal data in the Table(including 265 mesons and anti-mesons: 123 (u,d)-light mesons, 50 (s,u/d)-K mesons, 24 (c,u/d)-D mesons, 14 (s,c)-D mesons, 12 (b,u/d)-B mesons, 10 (s,b)-B_s mesons, 2 (c,b)-B_c mesons, 16 (c,$\bar{c}$) mesons, and 14 (b,$\bar{b}$) mesons). It is remarkable that among 265 mesons, 259 mesons are well described by this model within mass error less than 23%.

In addition, the radial wave-functions of mesons in configuration space can be obtained from the radial wave-functions in momentum space by Fourier transformation(see Appendix C), then one can calculate the mean square radii and the decay constants for some pseudoscalar mesons listed in Tab.I and compared with experimental data.

TABLE I: The mean square radii and decay constants of some pseudoscalar mesons, compared with the experimental data[29]. (Radii are given in fm^2 and decay constants are given in MeV)

	π^+	K^+	D^+	D_s	B
$\langle r^2\rangle_{the}$	0.385	0.253	0.235	-	-
$\langle r^2\rangle_{exp}$	0.452	0.314	-	-	-
f_{the}	135.2	210.7	189.2	253.1	227.5
f_{exp}	130.4	155.5	205.8	273	216

TABLE II: The pseudoscalar mesons mass spectra (in MeV).

Meson	$I^G(J^{PC})$	Exp(Mev)	Our's(Mev)	err(%)
π^0	0^{-+}	135	135	0
$\pi^\pm$	0^{-+}	140	140	0
η	0^{-+}	548	143	73
$\eta(958)$	0^{-+}	958	690	27
$\eta(1295)$	0^{-+}	1294	1258	2.8
$\pi(1300)^\pm$	0^{-+}	1300±100	1408	0
$\pi(1300)^0$	0^{-+}	1300±100	1350	0
$\eta(1405)$	0^{-+}	1410	1652	17
$\eta(1475)$	0^{-+}	1476	1700	15
$\eta(1760)$	0^{-+}	1756	1769	0.8
$\pi(1800)^\pm$	0^{-+}	1816	1454	19
$\pi(1800)^0$	0^{-+}	1816	2096	15.4
$X(1835)$	0^{-+}?	1833	2110	15.1
$\eta(2225)$	0^{-+}	2220	2160	2.7
$K^\pm$	0^-	494	494	0
K^0	0^-	498	494	0.8
$K(1460)$	0^-	1460	1522	4.2
$K(1830)$	0^-	1830	1597	12.7
D^0	0^-	1865	1931	3.5
$D^\pm$	0^-	1869	1931	3.3
$D_s^\pm$	0^-	1969	2001	1.6
B^0	0^-	5279	5584	5.8
$B^\pm$	0^-	5279	5584	5.8
B_s^0	0^-	5367	5667	5.6
$B_c^\pm$	0^-	6286	6342	0.9
$\eta_c(1S)$	0^{-+}	2980	2980	0
$\eta_c(2S)$	0^{-+}	3637	3533	2.9
$\eta_b(1S)$	0^{-+}	9391	8800	6.3

TABLE III: The scalar mesons mass spectra (in MeV).

Meson	J^{PC}	Exp(Mev)	Our's(Mev)	err(%)
$f_0(600)$	0^{++}	400-1200	736	0
$f_0(980)$	0^{++}	980	994	1.4
$a_0(980)^0$	0^{++}	985	1080	9.6
$a_0(980)^\pm$	0^{++}	985	930	5.6
$f_0(1370)$	0^{++}	1200-1500	1231	0
$a_0(1450)^0$	0^{++}	1474	1333	9.5
$a_0(1450)^\pm$	0^{++}	1474	1457	1.1
$f_0(1500)$	0^{++}	1505	1522	1.1
$f_0(1710)$	0^{++}	1724	1568	9.0
$f_0(2020)$	0^{++}	1992	1606	19.4
$f_0(2100)$	0^{++}	2103	1989	5.4
$f_0(2200)$	0^{++}	2189	2026	7.4
$f_0(2330)$	0^{++}	2321	2052	11.6
$K_0^*(800)$	0^+	672	731	8.8
$K_0^*(1430)$	0^+	1412	1535	8.7
$D_0^*(2400)^0$	0^+	2352	2254	4.2
$D_0^*(2400)^\pm$	0^+	2403	2254	6.2
$D_{s0}^*(2317)^\pm$	0^+	2317	2169	6.4
$\chi_{c0}(1P)$	0^{++}	3415	3352	1.8
$\chi_{b0}(1P)$	0^{++}	9860	9860	0
$\chi_{b0}(2P)$	0^{++}	10232	9990	2.3

TABLE IV: The axial vector meson mass spectra (in MeV).

Meson	J^{PC}	Exp(Mev)	Our's(Mev)	err(%)
$h_1(1170)$	1^{+-}	1170	1027	12.2
$b_1(1235)^0$	1^{+-}	1230	1127	8.4
$b_1(1235)^\pm$	1^{+-}	1229	1343	9.3
$a_1(1260)^0$	1^{++}	1230	1276	3.7
$a_1(1260)^\pm$	1^{++}	1230	1371	11.4
$f_1(1285)$	1^{++}	1281	1295	1.1
$h_1(1380)$	1^{+-}	1386	1301	6.1
$f_1(1420)$	1^{++}	1426	1311	8.0
$f_1(1510)$	1^{++}	1518	1419	6.5
$h_1(1595)$	1^{+-}	1594	1495	6.2
$a_1(1640)^0$	1^{++}	1647	1745	6.0
$a_1(1640)^\pm$	1^{++}	1647	1724	4.7
$K_1(1270)$	1^+	1273	1459	14.6
$K_1(1400)$	1^+	1402	1484	5.8
$K_1(1650)$	1^+	1650	1757	6.4
$D_1(2420)^0$	1^+	2422	2400	0.9
$D_1(2420)^\pm$	1^+?	2423	2400	0.9
$D_1(2430)^0$	1^+	2427	2425	0.1
$D_{S1}(2460)^\pm$	1^+	2460	2530	2.9
$D_{S1}(2536)^\pm$	1^+	2535	2549	0.6
$B_1(5721)^0$	1^+	5721	5666	1.0
$B_{S1}(5830)^0$	1^+	5829	5800	0.5
$\chi_{c1}(1p)$	1^{++}	3510	3504	0.2
$h_{c1}(1p)$	1^{+-}	3526	3509	0.5
$\chi_{b1}(1p)$	1^{++}	9892	10040	1.5
$\chi_{b1}(2p)$	1^{++}	10255	10040	2.1

TABLE V: The vector meson mass spectra (in MeV).

Meson	J^{PC}	Exp(Mev)	Our's(Mev)	err(%)
$\rho(770)^0$	1^{--}	775	1015	31
$\rho(770)^\pm$	1^{--}	775	1239	60
$\omega(782)$	1^{--}	783	1270	62
$\phi(1020)$	1^{--}	1019	1334	31
$\omega(1420)$	1^{--}	1425	1410	1.0
$\rho(1450)^0$	1^{--}	1465	1636	11.6
$\rho(1450)^\pm$	1^{--}	1465	1323	9.7
$\rho(1570)^0$	1^{--}	1570	1641	4.5
$\rho(1570)^\pm$	1^{--}	1570	1740	10.8

$\omega(1650)$	1^{--}	1670	1675	0.3
$\phi(1680)$	1^{--}	1680	1786	6.3
$\rho(1700)^0$	1^{--}	1720	1836	6.7
$\rho(1700)^\pm$	1^{--}	1700	1362	19.8
$\rho(1900)^0$	1^{--}	1909	1996	4.6
$\rho(1900)^\pm$	1^{--}	1909	1761	7.8
$\rho(2150)^0$	1^{--}	2149	2087	2.9
$\rho(2150)^\pm$	1^{--}	2149	2430	13.1
$K^*(892)$	1^-	892	1345	50.1
$K^*(1410)$	1^-	1414	1415	0.1
$K^*(1630)$	$1^-?$	1629	1502	7.8
$K^*(1680)$	1^-	1717	1531	10.8
$D^*(2007)^0$	1^-	2007	2100	4.6
$D^*(2010)^\pm$	1^-	2010	2126	5.8
$D^*(2640)$	$1^-?$	2637	2403	8.9
$D_S^{*\pm}$	$1^-?$	2112	2214	4.8
$D_{S1}(2700)^\pm$	1^-	2690	2233	16.9
B^*	1^-	5325	5518	3.6
B_S^*	1^-	5413	5625	3.9
J/ψ	1^{--}	3097	3284	6.0
$\psi(2S)$	1^{--}	3686	3362	8.8
$\psi(3770)$	1^{--}	3773	3684	2.3
$\psi(4040)$	1^{--}	4039	3700	8.4
$\psi(4160)$	1^{--}	4153	4197	1.1
$X(4260)$	1^{--}	4263	4769	11.9
$X(4360)$	1^{--}	4361	4783	9.7
$\psi(4415)$	1^{--}	4421	5341	20.8
$X(4660)$	1^{--}	4664	5418	16.2
$\gamma(1S)$	1^{--}	9460	9693	2.4
$\gamma(2S)$	1^{--}	10023	10013	0.1
$\gamma(3S)$	1^{--}	10355	10060	2.8
$\gamma(4S)$	1^{--}	10580	10765	1.7
$\gamma(10860)$	1^{--}	10865	10783	0.8
$\gamma(11020)$	1^{--}	11019	10861	1.4

TABLE VI: The tensor and pseudotensor meson mass (in MeV).

Meson	J^{PC}	Exp(MeV)	Theor(MeV)	err(%)
$\pi_2(1670)$	2^{-+}	1672	1587	5.1
$\pi_2(1880)$	2^{-+}	1895	1589	16.1
$\pi_2(2100)$	2^{-+}	2090	1922	8.0
$K_2(1580)$	2^-	1580	1530	3.2
$K_2(1770)$	2^-	1773	1539	13.2
$K_2(1820)$	2^-	1816	1763	2.9
$K_2(2250)$	2^-	2247	1765	21.4
$\gamma(1D)$	2^{--}	10161	10218	0.6
$a_2(1320)$	2^{++}	1318	1421	7.8
$a_2(1700)$	2^{++}	1723	1474	14.5
$K_2^*(1430)$	2^+	1425	1531	7.4
$K_2^*(1980)$	2^+	1973	1575	20.1
$D_2^*(2460)^\pm$	2^+	2460	2456	0.1
$D_2^*(2460)^0$	2^+	2462	2456	0.2
$D_{S2}^*(2573)^\pm$	$2^+?$	2573	2580	0.3
$B_J^*(5732)$	$2^+?$	5698	5706	0.1
$B_2^*(5747)^0$	2^+	5743	5765	0.4
$B_{S2}^*(5840)^0$	2^+	5840	5831	0.2
$B_{SJ}^*(5850)^0$	$2^+?$	5853	5883	0.5
$\chi_{c2}(1P)$	2^{++}	3556	3732	4.9
$\chi_{c2}(2P)$	2^{++}	3929	3745	4.7
$\chi_{b2}(1P)$	2^{++}	9912	10014	1.0
$\chi_{b2}(2P)$	2^{++}	10269	10354	0.8

TABLE VII: The mesons of $J \geq 3$ (in MeV).

Meson	J^{PC}	Exp(MeV)	Theor(MeV)	err(%)
$\omega_3(1670)$	3^{--}	1672	1677	0.3
$\rho_3(1690)^\pm$	3^{--}	1688	1702	0.8
$\rho_3(1690)^0$	3^{--}	1688	1697	0.5
$\phi_3(1850)$	3^{--}	1854	1807	2.5
$\rho_3(1990)^\pm$	3^{--}	1982	1795	9.4
$\rho_3(1990)^0$	3^{--}	1982	1807	8.8
$\rho_3(2250)^\pm$	3^{--}	2230	2660	19.2
$\rho_3(2250)^0$	3^{--}	2230	1852	17.0
$K_3^*(1780)$	3^-	2324	1777	23.5
$K_3(2320)$	3^+	2324	1812	22.0
$a_4(2040)^\pm$	4^{++}	2001	1745	12.7
$a_4(2040)^0$	4^{++}	2001	1743	12.9
$f_4(2050)$	4^{++}	2018	1865	7.6
$f_4(2300)$	4^{++}	2300	2016	12.3
$K_4^*(2045)$	4^+	2045	1827	10.6
$K_4(2500)$	4^-	2490	1933	22.3
$\rho_5(2350)^\pm$	5^{--}	2330	2292	1.6
$\rho_5(2350)^0$	5^{--}	2330	2218	4.8
$K_5^*(2380)$	5^-	2382	2352	1.3
$a_6(2450)^\pm$	6^{++}	2450 ± 130	2412	0
$a_6(2450)^0$	6^{++}	2450 ± 130	2423	0
$f_6(2510)$	6^{++}	2465 ± 50	2649	5.3

VI. ANALYSIS OF THE RESULTS

In the above calculations, only one set of parameters are used, which deserves discussion. The effective coupling strength or running coupling constant $\bar{\alpha}$ and the related constituent quark masses have a great influence on ground state of light mesons, such as π. The confining potential strength λ governs the quark confinement at large distances and has strong influence on the excited states of light mesons and also on the spectra of heavy mesons. From the recent experiments of hadron physics, we know that the QCD coupling $\alpha(Q^2)$ becomes large constant(not singular) in the low momentum limit, which is called infrared conformal invariance [30]. This experimental fact explains why our model with a set of constant parameters works well to describe the structures of mesons in the energy region of 0.14GeV$\rightarrow$10GeV, and our results may be thought of confirming the infrared conformal invariance feature of QCD on meson sector.

For light scalar mesons such as a_0 ,K_0^*, etc., although the structure of the scalar mesons remains a challenging puzzle, our model still describes $a_0(980)$, $a_0(1450)$, $K_0^*(800)$, etc. quite well. For heavy mesons, because of the large masses of heavy quarks, the effective double-gluon-exchange interactions for off-diagonal heavy mesons are weak, which makes the model applicable to them. Therefore, the calculated mass spectra for the mesons of $u/d\bar{s}$, $u/d\bar{c}$, $s\bar{c}$, $c\bar{c}$, $c\bar{b}$, $u/d\bar{b}$, $s\bar{b}$, and $b\bar{b}$ are in good agreement with the data. However the meson $K^*(892)$ on $u/d\bar{s}$ sector with larger error of 50.1% needs special investigation(see below).

It should be noted that the J and P of $D_s^{*\pm}$ are not identified by experiments, but their width and decay modes are observed and consistent with the 1^- state. Nevertheless, our model provides a definite assignment

of $J = 1$ and $P = -1$ for $D_s^{*\pm}$. A similar prediction of the unidentified J and P is also made for other 8 mesons: $X(1835)$, $D_1(2420)^{\pm}$, $K^*(1630)$, $D^*(2640)$, $D^*_{S2}(2573)^{\pm}$, $B^*_J(5732)$, $B^*_{SJ}(5850)^0$, and $f_J(2220)$.

The 6 mesons with errors larger than 23% provide some information. For the vector mesons of $\eta, \eta'(985), \rho(770)^0$, $\phi(1020)$, and $\omega(782)$ on u/d sector, and $K^*(892)$ on $(u/d)s$ sector , the large discrepancy indicates that the structures of these mesons are special than others and need a different set of parameters: indeed, as the set of parameters are re-adjusted to the set of ($\alpha = 0.4594, \gamma_0 = 0.58, \delta_0 = 0.74$) and with the others the same, a better fit is found with errors less than 23%. Increase of the effective interaction strengths implies that these vector mesons may have strong coupling between $q\bar{q}$ and $qq\bar{q}\bar{q}$ subspaces and among different flavor components.

VII. CONCLUSION AND DISCUSSION

In conclusion, we have formulated the QCD inspired relativistic bound state model for mesons and derived its mass eigen equations in total angular momentum representation. It is proved that in center of mass frame and in internal Hilbert subspace, total angular momentum of the meson system is conserved. Moreover, by taking the advantages of other effective QCD approaches [6, 26], the model has been improved significantly by introducing both a relativistic confining potential and an $SU(3)$ flavor mixing interaction. The resulting radial mass eigen equations are solved numerically and nonperturbatively, and 265 mesons including flavor off-diagonal mesons and flavor diagonal ones with $J = 0-6$ are calculated and compared with experimental data. The calculated masses are in good agreement with the data within the mean square root mass error of 14%, only 6 mesons with mass error larger than 23%. Besides, the wave functions obtained from the model also yield reasonable mean square radii and decay constants for some pseudo scalar mesons. In view that the structure of the light scalar mesons is still a subject of controversy[31], and the internal dynamics of heavy-light mesons in the static limit is far more complicated than that of the heavy-heavy ones[32], our model can be thought to be successful to describe a large body of mesons.

The comparison of our model with other approaches is as follows:

1. As Pauli's model is concerned, we have improved the model significantly on 5 important points and make it a predictive and systematic model for mesons: 1) Proving that in internal Hilbert subspace, total angular momentum is conserved; 2) establishing the mass eigen equations in total angular representation for the first time; 3) introducing the relativistic confining potential into the model, which is new and quite different from Pauli, and its form taken from the [6, 26]; 4) including the flavor mixing interaction; 5) solving the mass equations for 265 mesons nonperturbatively and the results are in good agreement with the data.

2. Comparing to other BSE and CQM meson models, our model is novel in following points: 1) The effective Hamiltonian is derived within the framework of light-front QCD and the form (the spinor structure) of the effective interactions is fixed by the lowest order of light-front QCD. 2) The mass eigen equation is for the squared rest mass, the separation between kinematical energy operator and interaction operators is rigorous. 3)The spinor structure of the effective interaction make it momentum-energy dependent. 4) Also due to the spinor structure of the effective interactions, the dynamics of spin-spin, spin-orbital, and tensor interactions (especially the spin singlet-triplet mxing and orbital angular momentum mixing) are included(see Appendix C,D). 5) The predictive power and the descriptive precision of the model are much better.

3. Comparing to holographic light-font QCD model of Brodsky *et al.*[10], our model has the following new aspects : 1) In the effective Hamiltonian of mesons, the kinematical energy operator is identical for both holographic light-front QCD model and our model, but the interaction terms are quite different. 2) Holographic light-front QCD model does not specify the effective interaction in detail, but just simulates confining potential by boundary condition (or harmonic oscillator potential), or recently by a positive-sign dilaton metric to generate confinement and break conformal symmetry; instead, our model provides a detailed semi-phenomenological effective interaction including its spinor structure, the confining potential, and the flavor mixing interactions. 3) Holographic light-front QCD model does not include spin-spin, spin-orbital, and tensor interactions, the total angular momentum of the system is not treated properly (although it has potential to describe the spin splittings); in the contrary, our model specifies the spin interactions and the spin dynamics is described fully in total angular momentum representation. 4) Finally, our model has been applied to a larger number of mesons (265 mesons identified experimentally) with higher precision than those of holographic light-front QCD model. In the above respects, our model has provided a tentative and effective solution to the problems listed above and the results are amazingly in good agreement with experimental data. In this sense, our model can be considered to be of complementarity to and refinement of the holographic light-front QCD model.

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Appendix A: Dynamics in light front form and instant form in center of mass frame and in internal Hilbert subspace

To avoid misunderstanding of light-front dynamics, we start from a discussion of full contents of dynamics for both instant form(IF) and light front form(LF). The content of dynamics should contain the following four aspects, we list them for both dynamics of instant form and dynamics of light form as follows.

1. Full contents of dynamics in instant form

1) Definition of time $x^0 = ct = t(c = 1)$

2) Hamiltonian (energy) operator is defined as the time translation operator:

$$i\hbar \frac{\partial}{\partial x^0} \sim \hat{P}^0 = \hat{H} = \hat{M} \tag{A1}$$

$\hat{M}$ is dynamical mass operator.

3) Dynamics

(i) Time evolution dynamics: equation of motion (Schrödinger equation),

$$i\hbar \frac{\partial \Psi}{\partial x^0} = \hat{H}\Psi = \hat{M}\Psi \tag{A2}$$

(ii) Stationary dynamics: for stationary solution ,

$$\Psi(t) = e^{-iMt/\hbar}\Psi \tag{A3}$$

one has the Hamiltonian eigen equation

$$\hat{H}\Psi = \hat{M}\Psi = M\Psi, \tag{A4}$$

where M is the eigen value of $\hat{M}$.

4) Specification of dynamical operators and kinematical operators among Poincare generators: 6 kinematical operators: $\hat{P}^i, \hat{J}^i, (i = 1, 2, 3)$; 4 dynamical operators: $\hat{P}^0, \hat{K}^i, (i = 1, 2, 3)$.

It should be noted that the dynamical operators contain interactions via the Hamiltonian and Lorentz boost operators while the kinematical operators do not. Consequently, the kinematical operators can be used to characterize the state of the system as good quantum numbers according their algebraic structure and the dynamical operators except the Hamiltonian operator can not play such a role. It should be emphasized that the above specification is made in whole Hilbert space of the states of composite systems. For a composite many-body system, the whole Hilbert space of states can be factorized into two parts: a) the center of mass motion characterized by its momentum $\vec{P}$, and b) the internal motion characterized by internal quantum numbers and $(J, J^3\ )$. Correspondingly, the Poincare operators contain two kinds of operations, one on the subspace of center of mass motion and the other on the subspace of internal motion. Since the center of mass motion can always be separated from the internal motion, the state wave function of the composite system Ψ can be written as $\Psi = \Psi_{cm}\Psi_{inter}$, where the wave function of center of mass motion is characterized by the center of mass momentum, namely $\Psi_{cm} = \Psi_{\vec{P}}$ with $\hat{P}^i\Psi_{\vec{P}} = P^i\Psi_{\vec{P}}$, while the internal wave function is characterized by internal quantum numbers and $(J, J^3\ )$.

2. Full contents of dynamics in light front form:

1) Definition of time x^+: $x^+ = x^0 + x^3$

2) Hamiltonian ("energy") operator is defined as the time translation operator:

$$i\hbar \frac{\partial}{\partial x^+} \sim \hat{P}^- \tag{A5}$$

From

$$\hat{P}^+\hat{P}^- - \hat{P}_\perp^2 = \hat{P}_0\hat{P}^0 - \hat{P}_3^2 - \hat{P}_1^2 - \hat{P}_2^2 = \hat{M}_0^2 \tag{A6}$$

$\hat{M}_0$ is rest mass operator; one has

$$\hat{P}^- = \frac{1}{\hat{P}^+}(\hat{M}_0^2 + \hat{P}_\perp^2) \tag{A7}$$

3) Dynamics

(i) Time evolution dynamics: equation of motion (Schrödinger equation),

$$i\hbar \frac{\partial}{\partial x^+}\Psi = \frac{1}{\hat{P}^+}(\hat{M}_0^2 + \hat{P}_\perp^2)\Psi \tag{A8}$$

(ii) Stationary dynamics: for stationary solution $\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp, x^+)$ with quantum numbers: "energy" $E^- = \frac{M^2}{P^+}$ and momentum $\vec{P} = (P^+, \vec{P}_\perp)(E^-$ is the eigen value of $\hat{P}^-$, P^+ and $\vec{P}_\perp$ are eigen values of $\hat{\vec{P}}^+$, $\hat{\vec{P}}_\perp$),

$$\Psi(x^+) = e^{-iM^2x^+/P^+\hbar}\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp) \tag{A9}$$

One has mass eigen equation:

$$\frac{M^2}{P^+}\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp) = \frac{1}{P^+}(\hat{M}_0^2 + \hat{P}_\perp^2)\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp) \tag{A10}$$

or

$$\hat{M}_0^2\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp) = (M^2 - \vec{P}_\perp^2)\Psi(\frac{M^2}{P^+}, P^+, \vec{P}_\perp) \tag{A11}$$

4) Specification of dynamical operators and kinematical operators among Poincare generators: 7 kinematical operators: $\hat{P}^+$, $\hat{J}^3$, $\hat{P}^i(i = 1, 2)$,$\hat{K}^3$, $\hat{E}^1 = \hat{K}^1 + \hat{J}^2$, $\hat{E}^2 = \hat{K}^2 - \hat{J}^1$; 3 dynamical operators: $\hat{P}^-$, $\hat{F}^1 = \hat{K}^1 - \hat{J}^2$, $\hat{F}^2 = \hat{K}^2 + \hat{J}^1$.

3. Dynamics in center of mass frame and in internal Hilbert subspace for both forms of dynamics

The internal structure of a composite system should be described in the rest frame as well as in the corresponding internal Hilbert subspace. Since the center of mass frame always follows the center of mass motion of the system and the position of the center of mass of the system is at the origin of the frame, the wave function of center of mass motion of the system should be $\Psi_{\vec{P}=0}$, and the center of mass momentum and the center of mass coordinates of the system should be zero, namely $<\Psi_{\vec{P}=0}|\hat{P}^i|\Psi_{\vec{P}=0}>=0$ and $<\Psi_{\vec{P}=0}|\hat{x}^i|\Psi_{\vec{P}=0}>=0$. In the center of mass frame, the Hilbert subspace of center of mass motion is frozen to $\Psi_{\vec{P}=0}$, the whole Hilbert space of states of the system is thus projected onto the corresponding internal Hilbert subspace Ψ_{inter}. Consequently, the dynamics of the composite system is reduced to the internal dynamics. Projecting onto the frozen center of mass wave function and integrating out the center of mass degrees of freedom, one obtain the Poincare operators in the internal subspace Ψ_{inter} as follows.

1) Four momentum and property of time in center of mass frame and in internal Hilbert subspace.

In center of mass frame, the wave function of center of mass motion is : $\Psi_{\vec{P}=0}$. The four momentum operator in internal Hilbert subspace can be obtained by projecting out the center of mass degrees of freedom (namely averaging over the center of mass wave function). Since $\hat{P}^0=\hat{H}=(\hat{M}_0^2+\hat{\vec{P}}^2)^{1/2}$ and $\hat{P}^i\Psi_{\vec{P}=0}=0$, in internal Hilbert subspace, one has

$$\hat{P}^i_{inter}=\left\langle\Psi_{\vec{P}=0}|\hat{P}^i|\Psi_{\vec{P}=0}\right\rangle=0 \tag{A12}$$

$$\hat{P}^0_{inter}=\left\langle\Psi_{\vec{P}=0}|\hat{P}^0|\Psi_{\vec{P}=0}\right\rangle=\hat{M}_0. \tag{A13}$$

Here $\hat{M}_0$ is the operator of rest mass of the system.

Thus in internal Hilbert subspace, the four momentum operators for instant form read:

$$\hat{P}^\mu_{inter}=(\hat{M}_0,0,0,0) \tag{A14}$$

while four momentum operators for light front form are:

$$\hat{P}^\mu_{inter}=(\hat{M}_0,0,0,\hat{M}_0), \tag{A15}$$

From the above results , one has

$$\hat{P}^-_{inter}=\hat{P}^0_{inter}=\hat{P}^+_{inter}=\hat{M}_0, \tag{A16}$$

$$i\hbar\frac{\partial}{\partial x^+}=i\hbar\frac{\partial}{\partial x^0}=i\hbar\frac{\partial}{\partial\tau}, \tag{A17}$$

where τ is the proper time corresponding to the rest mass operator $\hat{M}_0$. The last equation leads to

$$x^+=\tau+\tau_0,\ x^0=\tau+\tau_0' \tag{A18}$$

where τ_0 and τ_0' are constant shifts of proper time. One can choose the start point of time such that

$$\tau=0\to x^+=x^0=0 \tag{A19}$$

This leads to

$$\tau_0=\tau_0'=0 \tag{A20}$$

and

$$x^+=x^0=\tau \tag{A21}$$

2) Dynamics in center of mass frame and in internal Hilbert subspace

(i) Time evolution dynamics: equations of motion in center of mass frame and in internal Hilbert subspace

The Schrödinger equations

$$i\hbar\frac{\partial\Psi}{\partial x^+}=\hat{M}_0\Psi \tag{A22}$$

in light front form, and

$$i\hbar\frac{\partial\Psi}{\partial x^0}\Psi=\hat{M}_0\Psi \tag{A23}$$

in instant form become the same

$$i\hbar\frac{\partial\Psi}{\partial\tau}=\hat{M}_0\Psi \tag{A24}$$

(ii) Stationary dynamics: mass (energy) eigen equations in center of mass frame and in internal Hilbert subspace.

The mass eigen equations

$$\hat{M}_0\Psi=M_0\Psi \tag{A25}$$

in instant form where M_0 is the eigen value of $\hat{M}_0$, and

$$\hat{M}_0^2\Psi=M_0^2\Psi \tag{A26}$$

in light front form are also the same because multiplying $\hat{M}_0$ on the first equation leads to the second one.

3) Kinematical and dynamical operators in center of mass frame and in internal Hilbert subspace

Projecting onto internal Hilbert subspace, the kinematical and dynamical operators can be obtained from the following calculation. From the results of (A12-A15) and

$$\hat{J}^i=\hat{J}^i_{cm}+\hat{J}^i_{inter}, \tag{A27}$$

$$\hat{J}^1_{cm}=\hat{x}^2\hat{P}^3-\hat{P}^2\hat{x}^3, cyclic, \tag{A28}$$

one obtain

$$\left\langle\Psi_{\vec{P}=0}|\hat{J}^i_{cm}|\Psi_{\vec{P}=0}\right\rangle=0, \tag{A29}$$

$$\left\langle\Psi_{\vec{P}=0}|\hat{J}^i|\Psi_{\vec{P}=0}\right\rangle=\hat{J}^i_{inter} \tag{A30}$$

$$\left\langle \Psi_{\vec{P}=0}|\hat{K}^i|\Psi_{\vec{P}=0}\right\rangle = \left\langle \Psi_{\vec{P}=0}|\hat{x}^0\hat{P}^i - \hat{x}^i\hat{P}^0|\Psi_{\vec{P}=0}\right\rangle \\ = \langle \Psi_{\vec{P}=0}|\hat{x}^i|\Psi_{\vec{P}=0}\rangle \hat{M}_0 = 0, \tag{A31}$$

$$\left\langle \Psi_{\vec{P}=0}|\hat{E}^1|\Psi_{\vec{P}=0}\right\rangle = \left\langle \Psi_{\vec{P}=0}|\hat{K}^1 + \hat{J}^2|\Psi_{\vec{P}=0}\right\rangle = \hat{J}^2_{inter} \tag{A32}$$

$$\left\langle \Psi_{\vec{P}=0}|\hat{E}^2|\Psi_{\vec{P}=0}\right\rangle = \left\langle \Psi_{\vec{P}=0}|\hat{K}^2 - \hat{J}^1|\Psi_{\vec{P}=0}\right\rangle = -\hat{J}^1_{inter} \tag{A33}$$

$$\left\langle \Psi_{\vec{P}=0}|\hat{F}^1|\Psi_{\vec{P}=0}\right\rangle = \left\langle \Psi_{\vec{P}=0}|\hat{K}^1 - \hat{J}^2|\Psi_{\vec{P}=0}\right\rangle = -\hat{J}^2_{inter} \tag{A34}$$

$$\left\langle \Psi_{\vec{P}=0}|\hat{F}^2|\Psi_{\vec{P}=0}\right\rangle = \left\langle \Psi_{\vec{P}=0}|\hat{K}^2 + \hat{J}^1\hat{P}^0|\Psi_{\vec{P}=0}\right\rangle = \hat{J}^1_{inter} \tag{A35}$$

From the above results, one obtains the same reduced and degenerated kinematical and dynamical operators for both forms of dynamics in internal Hilbert subspace as follows: kinematical operators: $\hat{J}^i_{inter}(i = 1, 2, 3)$; dynamical operator: $\hat{P}^0_{inter} = \hat{P}^-_{inter} = \hat{M}_0$.

The above results tell that in center of mass frame and in internal Hilbert subspace, light front time and instant time, light front dynamics and instant dynamics, light front angular momentum and instant angular momentum are identical.

4. Conclusion

In general frames and in whole Hilbert space, both forms of dynamics are quite different. However, in center of mass frame and in internal Hilbert subspace, the two forms of dynamics are reduced to the identical internal dynamics.

There is a dilemma in this paper at first glance: our model begins with a light front QCD model, but the final form of our model possesses the feature of instant dynamics of QCD. Is it of LF dynamics or IF dynamics? The solution to the dilemma is given in this Appendix, the answer is that in center of mass frame and in internal Hilbert subspace, the reduced internal dynamics of both forms are identical.

Therefore, our model contains ingredients of both the instant form and light front form of QCD, it can be called as QCD inspired effective Hamiltonian meson model.

Appendix B: Conservation of total angular momentum in internal Hilbert subspace

The reduction of angular momentum operators in internal Hilbert subspace can be discussed in an alternative manner and the results are the same as that in Appendix A.

A relativistic dynamical system has inhomogeneous Lorentz symmetry defined by the *Poincaré algebra*: P^μ is energy-momentum vector, and $M^{\mu\nu}$ is used to describes the rotational and boost transformations. In instant form, the angular momentum and boost vectors are given as: $M^{ij} = \epsilon_{ijk}J^k$ and $M^{0i} = K^i$

Now define the "quasi angular momentum" operators in the light-front form:

$$\mathcal{J}^3 = J^3 + \frac{\varepsilon_{ij}\boldsymbol{E}^i_\perp \boldsymbol{P}^j_\perp}{P^+} \ , \\ \mathcal{J}^{\perp i} = M_0^{-1}\varepsilon_{ij}(\frac{1}{2}(\boldsymbol{F}^j_\perp P^+ - \boldsymbol{E}^j_\perp P^-) - K^3\boldsymbol{P}^j_\perp \\ + \mathcal{J}^3\varepsilon_{jl}\boldsymbol{P}^l_\perp), (i, j = 1, 2). \tag{B1}$$

It is easy to prove that they satisfy the SU(2) algebra:

$$[\mathcal{J}^i, \mathcal{J}^j] = i\epsilon_{ijk}\mathcal{J}^k \tag{B2}$$

It is very useful to define a 'light-front Hamiltonian' as the operator:

$$H_{LC} = P^\mu P_\mu = P^-P^+ - \vec{P_\perp}^2 = \hat{M}_0^2 \tag{B3}$$

H_{LC} commutes with the quasi angular momentum operators :

$$[H_{LC}, \vec{\mathcal{J}}] = 0. \tag{B4}$$

In principle, one could label the eigen states as $|M, P^+, \vec{P_\perp}, \vec{\mathcal{J}}^2, \vec{\mathcal{J}_3}\rangle$, since $\mathcal{J}_3$ is kinematical. However, $\vec{\mathcal{J}_\perp}$ is dynamical and depends on the interactions. Thus it is generally difficult to explicitly compute the total spin $\vec{\mathcal{J}}$ of a state using light-front quantization. Fortunately, in center-of-mass frame and in internal Hilbert subspace, by using the results of Appendix A, one has the following equations,

$$\left\langle \Psi_{\vec{P}=0}|\mathcal{J}^3|\Psi_{\vec{P}=0}\right\rangle = J^3_{inter} \ , \\ \left\langle \Psi_{\vec{P}=0}|\mathcal{J}^i|\Psi_{\vec{P}=0}\right\rangle = J^i_{inter} \\ (i, j = 1, 2) \ . \tag{B5}$$

Therefore, in internal Hilbert subspace, the quasi angular momentum operators $\mathcal{J}^i_{inter}$ are identical to the total angular momentum operators $J^i_{inter}(i = 1, 2, 3)$, the total angular momentum is conserved, and the eigen equation of the Hamiltonian H_{LC} of the internal dynamics can be solved in the total angular momentum representation.

Appendix C: Derivation of the radial mass eigen equations in total angular momentum representation

According to Pauli et al., the effective mass eigen equation of mesons of light-front QCD in center of mass frame and in internal Hilbert subspace reads:

$$\left[M_0^2-(E_1(k)+E_2(k))^2\right]\varphi_{s_1s_2}(\boldsymbol{k}) = \sum_{s_1's_2'}\int d^3\boldsymbol{k}U_{s_1s_2;s_1's_2'}(\boldsymbol{k};\boldsymbol{k'})\varphi_{s_1's_2'}(\boldsymbol{k'}), \quad (C1)$$

where

$$U_{s_1s_2;s_1's_2'} = \frac{4m_s}{3\pi^2}\frac{\overline{\alpha}(Q)}{Q^2}R(Q)\frac{S_{s_1s_2;s_1's_2'}}{\sqrt{A(k)A(k')}} \quad (C2)$$

with

$$\begin{aligned} S_{s_1s_2;s_1's_2'} &= [\overline{u}(\boldsymbol{k},s_1)\gamma_\mu(1)u(\boldsymbol{k'},s_1')] \\ &\times [\overline{v}(-\boldsymbol{k},s_2)\gamma^\mu(2)v(-\boldsymbol{k'},s_2')] \end{aligned} \quad (C3)$$

and

$$\begin{aligned} \frac{1}{A(k)} &= m_r\left(\frac{1}{E_1(k)}+\frac{1}{E_2(k)}\right), \\ m_s &= m_1+m_2,\ m_r=\frac{m_1m_2}{m_1+m_2}, \\ Q &= Q(\boldsymbol{k};\boldsymbol{k'}). \end{aligned} \quad (C4)$$

Equation (A1) can be written as Schrödinger equation in the light front QCD,

$$\widehat{H}\Psi_{\rm meson} = M_0^2\Psi_{\rm meson} \quad (C5)$$

The general eigen wave function $\Psi_{\rm meson}$ of meson can be expressed in momentum-spin representation,

$$\Psi_{\rm meson} = \sum_{s_1,s_2}\int d^3\boldsymbol{k}\varphi_{s_1s_2}(\boldsymbol{k})\,|\chi(s_1)\chi(s_2)\cdot\boldsymbol{k}\rangle\,. \quad (C6)$$

Here basis of the momentum-spin representation are

$$\langle\boldsymbol{r}|\chi(s_1)\chi(s_2)\cdot\boldsymbol{k}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\chi(s_1)\chi(s_2)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \quad (C7)$$

where the spin wave functions and their orthogonal conditions read

$$\chi(+\frac{1}{2}) = \begin{pmatrix}1\\0\end{pmatrix},\ \chi(-\frac{1}{2}) = \begin{pmatrix}0\\1\end{pmatrix}, \quad (C8)$$

$$\langle\chi(s_1)|\chi(s_2)\rangle = \delta_{s_1s_2}. \quad (C9)$$

The orthogonal conditions of the spinors are

$$\begin{aligned} &\langle\boldsymbol{k}\cdot\overline{u}(\boldsymbol{k},s_1)\overline{v}(-\boldsymbol{k},s_2)|u(\boldsymbol{k'},s_1')v(-\boldsymbol{k'},s_2')\cdot\boldsymbol{k'}\rangle \\ &\quad = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})\delta_{s_1s_1'}\delta_{s_2s_2'}, \quad (C10) \\ &\overline{u}(\boldsymbol{k},s_1)u(\boldsymbol{k},s_1') = \delta_{s_1s_1'}, \quad (C11) \\ &\overline{v}(-\boldsymbol{k},s_2)v(-\boldsymbol{k},s_2') = \delta_{s_2s_2'}, \quad (C12) \end{aligned}$$

and the completeness conditions read,

$$\sum_s u(\boldsymbol{k},s)\overline{u}(\boldsymbol{k},s) = \frac{1}{2m}\left(\gamma_\mu k_1^\mu+m\right), \quad (C13)$$

$$\sum_s v(-\boldsymbol{k},s)\overline{v}(-\boldsymbol{k},s) = \frac{1}{2m}\left(\gamma_\mu k_2^\mu-m\right), \quad (C14)$$

where $k_1^\mu=(E_1(k),\boldsymbol{k})$, and $k_2^\mu=(E_2(k),-\boldsymbol{k})$.

According to the Dirac form of quantum mechanics, in the eigen equation (C5), the Dirac form of the Hamiltonian operator is

$$\widehat{H} = \widehat{E}+\widehat{U}, \quad (C15)$$

where

$$\begin{aligned} \widehat{E} &= \int d^3\boldsymbol{k}\,[E_1(k)+E_2(k)]^2 \\ &\times\sum_{s_1s_2}|\chi(s_1)\chi(s_2)\cdot\boldsymbol{k}\rangle\langle\boldsymbol{k}\cdot\chi(s_1)\chi(s_2)|, \end{aligned} \quad (C16)$$

and

$$\begin{aligned} \widehat{U} = &\int d^3\boldsymbol{k}d^3\boldsymbol{k'}\sum_{s_1s_2;s_1's_2'}U(k,k') \\ &\times[\overline{u}(\boldsymbol{k},s_1)\overline{v}(-\boldsymbol{k},s_2)(\gamma_\mu(1)\gamma^\mu(2))u(\boldsymbol{k'},s_1')v(-\boldsymbol{k'},s_2')] \\ &\times|\chi(s_1)\chi(s_2)\cdot\boldsymbol{k}\rangle\langle\boldsymbol{k'}\cdot\chi(s_1')\chi(s_2')|, \end{aligned} \quad (C17)$$

with the definition,

$$U(k,k') \equiv \frac{1}{3m_r\pi^2}\frac{\overline{\alpha}(Q)}{Q^2}R(Q)\frac{1}{\sqrt{A(k)A(k')}}. \quad (C18)$$

In the above equation, as done by Pauli et al.[20], the light front $k-$ space has been transformed back to the Lab $k-$ space by the Terent'ev transformation, and Lepage-Brodsky (helicity) spinors have been transformed to the Bjorken-Drell (spin) spinors.

Using eqs.(C6, C15-C18) and projecting equation (C5) onto the subspace $|\chi(s_1)\chi(s_2)\cdot\boldsymbol{k}\rangle$, we recover the equation (C1), indicating that the Dirac Form of the eigen equation (C5) is equivalent that of (C1).

Since $(E_1(k)+E_2(k))^2$ and the interaction kernal operator $\widehat{U}[\boldsymbol{k},\boldsymbol{k'};\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2)]$ are scalar (see Appendix D, discussion below eq.(D9)), $\widehat{H}$ is rotational invariant with respect to the total angular momentum $\boldsymbol{J}_i=\boldsymbol{l}_i+\boldsymbol{s}_i^1+\boldsymbol{s}_i^2=\boldsymbol{l}_i+\boldsymbol{s}_i$, $[\widehat{H},\boldsymbol{J}_i]=0$. That means the total angular momentum $\widehat{\boldsymbol{J}}^2$ and $\widehat{J}_z$ are conserved. Based on this point, the wave function of the meson system can be written in total angular representation as follows,

$$\Psi_{meson}(k,\Omega_k,s) = \sum_{J,M}\sum_{l=|J-s|}^{J+s}\sum_{s=0,1}R_{Jsl}(k)\Phi_{JslM}(\Omega_k,s), \quad (C19)$$

were the total angular momentum eigen functions Φ_{JslM} of $\{\widehat{\boldsymbol{J}}^2,\widehat{J}_z,\widehat{\boldsymbol{s}}^2,\widehat{\boldsymbol{l}}^2\}$ are,

$$\Phi_{JslM}(\Omega_k,s) = \sum_{m\mu}\langle lms\mu|JM\rangle Y_{lm}(\Omega_k)\chi_{s\mu}(12), \quad (C20)$$

the eigen wave functions of spin singlet and triplet read as,

$$\chi_{s\mu}(12) = \sum_{s_1s_2}\langle\tfrac{1}{2}s_1\tfrac{1}{2}s_2|s\mu\rangle\chi(s_1)\chi(s_2). \quad (C21)$$

By virtue of the Fourier transformation in spherical coordinates, from the eigen wave function in the momentum radial $k-$space, one can obtain the corresponding wave function in the configuration radial $r-$ space,

$$\begin{aligned}
\Psi_{JM}(r,\Omega_r,s) &= \int d\boldsymbol{k}^3 \Psi_{JM}(k,\Omega_k,s) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\
&= \sum_{l=|J-s|}^{J+s} \sum_{s=0,1} \sum_{l',m'} \int k^2 dk R_{Jsl}(k) J_l(kr) \\
& \quad \int d\Omega_k \Phi_{JslM}(\Omega_k,s) Y^*_{l'm'}(\Omega_k) Y_{l'm'}(\Omega_r) \\
&= \sum_{l=|J-s|}^{J+s} \sum_{s=0,1} R_{Jsl}(r) \Phi_{JslM}(\Omega_r,s), \quad \text{(C22)}
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{JslM}(\Omega_r,s) &= \sum_{m\mu} \langle lms\mu|JM\rangle Y_{lm}(\Omega_r)\chi_{s\mu}(12), \\
R_{Jsl}(r) &= \int k^2 dk R_{Jsl}(k) J_l(kr), \\
J(kr) &= \sqrt{4\pi(2l+1)}\; i^l \; j_l(kr). \qquad \text{(C23)}
\end{aligned}$$

$j_l(kr)$ is the spherical Bessel function of order l.

Using the expression (C19) of the wave function Ψ_{meson}, projecting the mass eigen equation (C5) onto the Φ_{JslM} subspace from the left, and integrating out the spin and angular part of the wave function, we obtain the eigen equations for the radial wave functions $R_{Jsl}(k)$,

$$\begin{aligned}
& \left[M_0^2 - (E_1(k)+E_2(k))^2\right] R_{Jsl}(k) \qquad \text{(C24)} \\
&= \sum_{l'=|J-s'|}^{J+s'} \sum_{s'=0,1} \int k'^2 dk' U^J_{sl;s'l'}(k;k') R_{Js'l'}(k'),
\end{aligned}$$

where the kernel $U^J_{Sl;S'l'}(k;k')$ is defined as,

$$\begin{aligned}
& U^J_{sl;s'l'}(k;k') = \sum_{mm'} \sum_{s_1 s_2} \sum_{s'_1 s'_2} \int\int d\Omega_k d\Omega_{k'} \\
& \times \langle Y_{lm}(\Omega_k)| U_{s_1 s_2;s'_1 s'_2}(\mathbf{k},\mathbf{k}')| Y_{l'm'}(\Omega_{k'})\rangle \qquad \text{(C25)} \\
& \times \langle lms\mu|JM\rangle \langle \tfrac{1}{2}s_1 \tfrac{1}{2}s_2|s\mu\rangle \langle l'm's'\mu'|JM\rangle \langle \tfrac{1}{2}s'_1 \tfrac{1}{2}s'_2|s'\mu'\rangle.
\end{aligned}$$

This is a set of coupled equations for the radial functions $R_{Jsl}(k)$ that have different partial waves, spin singlet and triplet coupled by the tensor potentials and by the relativistic spin-orbital potential (see below).

Appendix D: Calculation of the interaction kernel in total angular momentum representation

The quark and anti-quark spinors are given in the Bjørken-Drell representation,

$$u(\boldsymbol{k}, s=+\tfrac{1}{2}) = \frac{1}{\sqrt{2m_1(E_1+m_1)}} \begin{pmatrix} E_1+m_1 \\ 0 \\ k_z \\ k_l \end{pmatrix},$$

$$u(\boldsymbol{k}, s=-\tfrac{1}{2}) = \frac{1}{\sqrt{2m_1(E_1+m_1)}} \begin{pmatrix} 0 \\ E_1+m_1 \\ k_r \\ -k_z \end{pmatrix},$$

$$v(-\boldsymbol{k}, s=+\tfrac{1}{2}) = \frac{1}{\sqrt{2m_2(E_2+m_2)}} \begin{pmatrix} -k_z \\ -k_l \\ E_2+m_2 \\ 0 \end{pmatrix},$$

$$v(-\boldsymbol{k}, s=-\tfrac{1}{2}) = \frac{1}{\sqrt{2m_2(E_2+m_2)}} \begin{pmatrix} -k_r \\ k_z \\ 0 \\ E_2+m_2 \end{pmatrix},$$

where

$$\begin{aligned}
k_{l,r} &= k_x \pm i k_y = k\sin\theta_k e^{\pm i\varphi_k} = k\sqrt{\frac{8\pi}{3}} Y_{1\pm1}(\theta_k,\varphi_k), \\
k_z &= k\cos\theta_k = k\sqrt{\frac{4\pi}{3}} Y_{10}(\theta_k,\varphi_k). \qquad \text{(D1)}
\end{aligned}$$

Defining the spherical spinors

$$\Phi^A_{\frac{1}{2}s}(\Omega_k) = \sum_{m\nu} \langle 1m\tfrac{1}{2}\nu|\tfrac{1}{2}s\rangle Y_{00}(\Omega_k)\chi(\nu) = \frac{1}{\sqrt{4\pi}}\chi(s), \qquad \text{(D2)}$$

$$\Phi^B_{\frac{1}{2}s}(\Omega_k) = \sum_{m\nu} \langle 1m\tfrac{1}{2}\nu|\tfrac{1}{2}s\rangle Y_{1m}(\Omega_k)\chi(\nu) = \frac{1}{\sqrt{4\pi}}\sigma_k\chi(s), \qquad \text{(D3)}$$

where $\sigma_k = (\boldsymbol{\sigma}\cdot\boldsymbol{k})/k$ and $\Omega_k = (\theta_k,\varphi_k)$($\sigma_k$ is pseudo scalar), the spinors can be re-expressed as

$$\begin{aligned}
u(\boldsymbol{k},s) &= \begin{pmatrix} A_1(k)\Phi^A_{\frac{1}{2}s}(\Omega_k) \\ B_1(k)\Phi^B_{\frac{1}{2}s}(\Omega_k) \end{pmatrix}, \qquad \text{(D4)} \\
v(-\boldsymbol{k},s) &= \begin{pmatrix} -B_2(k)\Phi^B_{\frac{1}{2}s}(\Omega_k) \\ A_2(k)\Phi^A_{\frac{1}{2}s}(\Omega_k) \end{pmatrix}, \qquad \text{(D5)}
\end{aligned}$$

where

$$A_i(k) = \sqrt{\frac{2\pi(E_i+m_i)}{m_i}},\; B_i(k) = \sqrt{\frac{2\pi k^2}{m_i(E_i+m_i)}}\,. \qquad \text{(D6)}$$

The spin factor $S_{s_1 s_2;s'_1 s'_2}$ of the interaction can be written as

$$
\begin{aligned}
S_{s_1s_2;s_1's_2'} &= [\overline{u}(\boldsymbol{k},s_1)\gamma_0(1)u(\boldsymbol{k'},s_1')]\,[\overline{v}(-\boldsymbol{k},s_2)\gamma_0(2)v(-\boldsymbol{k'},s_2')] - [\overline{u}(\boldsymbol{k},s_1)\gamma_i(1)u(\boldsymbol{k'},s_1')]\,[\overline{v}(-\boldsymbol{k},s_2)\gamma_i(2)v(-\boldsymbol{k'},s_2')]\\
&= \Big[A_1^*(k)A_1(k') + B_1^*(k)B_1(k')\big\langle\Phi^B_{\frac{1}{2}s_1}(\Omega_k)\big|\Phi^B_{\frac{1}{2}s_1'}(\Omega_{k'})\big\rangle\Big]\,\Big[A_2^*(k)A_2(k') + B_2^*(k)B_2(k')\big\langle\Phi^B_{\frac{1}{2}s_2}(\Omega_k)\big|\Phi^B_{\frac{1}{2}s_2'}(\Omega_{k'})\big\rangle\Big]\\
&\quad + \Big[A_1^*(k)B_1(k')\big\langle\Phi^A_{\frac{1}{2}s_1}(\Omega_k)\sigma_i\big|\Phi^B_{\frac{1}{2}s_1'}(\Omega_{k'})\big\rangle + B_1^*(k)B_1(k')\big\langle\Phi^B_{\frac{1}{2}s_1}(\Omega_k)\big|\sigma_i\Phi^A_{\frac{1}{2}s_1'}(\Omega_{k'})\big\rangle\Big]\\
&\quad \times \Big[B_2^*(k)A_2(k')\big\langle\Phi^B_{\frac{1}{2}s_1}(\Omega_k)\big|\sigma_i\Phi^A_{\frac{1}{2}s_1'}(\Omega_{k'})\big\rangle + A_2^*(k)B_2(k')\big\langle\Phi^A_{\frac{1}{2}s_2}(\Omega_k)\sigma_i\big|\Phi^B_{\frac{1}{2}s_2'}(\Omega_{k'})\big\rangle\Big]\\
&= \frac{1}{\sqrt{4\pi}}\Big\langle\chi(s_1)\chi(s_2)\Big|\Big\{\,[A_1^*(k)A_1(k') + B_1^*(k)B_1(k')\sigma_k(1)\sigma_{k'}(1)]\,[A_2^*(k)A_2(k') + B_2^*(k)B_2(k')\sigma_k(2)\sigma_{k'}(2)]\\
&\qquad\qquad + [A_1^*(k)B_1(k')\boldsymbol{\sigma}(1)\sigma_{k'}(1) + B_1^*(k)A_1(k')\sigma_k(1)\boldsymbol{\sigma}(1)]\\
&\qquad\qquad \cdot[B_2^*(k)A_2(k')\sigma_k(2)\boldsymbol{\sigma}(2) + A_2^*(k)B_2(k')\boldsymbol{\sigma}(2)\sigma_{k'}(2)]\,\Big\}\Big|\chi(s_1')\chi(s_2')\Big\rangle. \qquad \text{(D7)}
\end{aligned}
$$

The kernel $U^J_{sl;s'l'}(k;k')$ can be rewritten as

$$
U^J_{sl;s'l'}(k;k') = \big\langle\Phi_{JslM}(\Omega_k,s)\big|\widehat{U}[\boldsymbol{k},\boldsymbol{k}';\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2)]\big|\Phi_{Js'l'M}(\Omega_{k'},s')\big\rangle, \qquad \text{(D8)}
$$

where the interaction operator in momentum and spin space is

$$
\begin{aligned}
\widehat{U}[\boldsymbol{k},\boldsymbol{k}';\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2)] = \frac{U(k,k')}{\sqrt{4\pi}}\Big\{\,&[A_1^*(k)A_1(k') + B_1^*(k)B_1(k')\sigma_k(1)\sigma_{k'}(1)]\,[A_2^*(k)A_2(k') + B_2^*(k)B_2(k')\sigma_k(2)\sigma_{k'}(2)]\\
&+ [A_1^*(k)B_1(k')\boldsymbol{\sigma}(1)\sigma_{k'}(1) + B_1^*(k)A_1(k')\sigma_k(1)\boldsymbol{\sigma}(1)]\\
&\cdot[B_2^*(k)A_2(k')\sigma_k(2)\boldsymbol{\sigma}(2) + A_2^*(k)B_2(k')\boldsymbol{\sigma}(2)\sigma_{k'}(2)]\,\Big\}. \qquad \text{(D9)}
\end{aligned}
$$

Since σ_k and $\sigma_{k'}$ are pseudo scalar, k, k', σ_k $\sigma_{k'}$, and $\boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2)$ are scalar, the above interaction kernel operator $\widehat{U}[\boldsymbol{k},\boldsymbol{k}';\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2)]$ is scalar.

From the last expression of the kernel $U^J_{sl;s'l'}(k;k')$, we could see that the first term contributes to different kinds of central potentials and relativistic spin-orbit coupling potentials, the second term contributes to the tensor potentials changing l by $\Delta l = \pm 2$ and mixing spin singlet and triplet.

If $m_1 = m_2$ and the tensor potentials are neglected, l and s are conserved and the interaction kernel becomes diagonal in l and s representation,

$$
U^J_{sl;s'l'}(k;k') = U^J_{sl;sl}(k;k')\delta_{ll'}\delta_{ss'} = U_{Jsl}(k;k')\delta_{ll'}\delta_{ss'}. \qquad \text{(D10)}
$$

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